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Mirrorless lasing action via nonlinear interaction between photons and bosons

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Abstract. We study the lasing action due to nonlinear interactions of photons with boson elementary excitation, such as phonons and excitons, and point out the distinct features from the conventional laser model described by the two-level model. External feedback by means of cavity mirrors is unnecessary for the lasing action as well as for the increasing absorption optical bistability. It is shown that the Landau theory of the second-order phase transition is applicable to the threshold behaviour of the present mirrorless laser model.

1. Introduction

The two-level model plays an important role in the research of the interaction of radiation with condensed matter [1–6]. The conventional laser theory (Sargent *et al* [1], Haken [2] and Louisell [5]) and the theory of optical bistability (Gibbs [4], Bonifacio and Lugiato [6]) are based on the two-level model. However, the optical nonlinearity due to the coupling between photon and two-level model possesses the character of saturated absorption (SA), so this model cannot describe the mechanism for increasing absorption optical bisability (IAOB). Planck, in his explanation of thermal radiation, proposed a particle model for electromagnetic waves which was the beginning of quantum theory. As ideal black-body radiation is independent of the radiated object and dependent only on its temperature, Planck treated the black-body as a system consisting of 'a large number of similar simple periodic oscillators isolated from one another' [7], the dynamics was simplified: the oscillators emit and absorb photons; the excited states of the oscillators can be described by boson quasi-particle excitation in condensed media (e.g. phonons or excitons). In the light of Planck's model, one of the present authors has dealt with IAOB [8]. It has been demonstrated that the nonlinear interaction of photons with the boson elementary excitation can provide a novel IAOB mechanism. This paper will further study the lasing action due to this coupling.

The paper is organized as follows. We shall develop the Hamiltonian of photon–boson coupling in section 2. Coherent injection and incoherent injection will be discussed in

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section 3. In sections 4, 5 and 6 we shall solve the laser equations and discuss the stability, threshold behaviour of the laser and parameters of coupling; we shall give a concise conclusion in section 7.

2. The Hamiltonian of the coupling between photons and material bosons

The Hamiltonian of the system of photons and bosons in a single mode form is

$$H = \hbar\omega_\epsilon a^+ a + \hbar\omega_e b^+ b + \mathrm{i}\hbar g b^+(1 + \lambda b^+ b)a + \mathrm{HC} \qquad \lambda = g_{NL}/g \tag{1}$$

where (a^+, a) represent operators of the photon mode with frequency ω_ϵ, (b^+, b) stand for the operators of boson mode with frequency ω_e, which obey the boson commutation relation as photon operators,

$$[b, b^+] = 1 \tag{2}$$

where g is a linear coupling constant between photons and bosons. The nonlinear coupling constant $g_{NL}(= \lambda g)$, generally speaking, may be positive or negative, which depends on working materials, but the positive one is required for laser action, which will be shown later. By considering the polarization of electromagnetic interaction in the rotating wave approximation (RWA), it has been demonstrated that the boson mode in the Hamiltonian expressed by equation (1) may be either the optical phonons in ionic crystals [9] or the Wannier excitons in low-density case in semiconductors [10]. We shall present our results in parallel with the case of the two-level model, so we rewrite equation (1) as

$$\begin{aligned} &H = \hbar\omega_\epsilon a^+ a + \hbar\omega_e b^+ b + \mathrm{i}\hbar g(\pi^+ a - \pi a^+) \\ &\pi^+ = b^+(1 + \lambda b^+ b) \qquad \pi = (1 + \lambda b^+ b)b \end{aligned} \tag{3}$$

and (π^+, π) should be interpreted as the operators of matter polarization, as shown in [9]. When the operators $(b^+ b, \pi^+, \pi)$ in equation (3) are replaced by the Pauli operators $(\sigma_3, \sigma_+, \sigma_-)$, equation (3) becomes the well known Hamiltonian of photons interacting with a two-level system [1, 2, 5]

$$H = \hbar\omega_\epsilon a^+ a + \hbar\omega_e \sigma_3 + \mathrm{i}\hbar g(\sigma_+ a - \sigma_- a^+) \tag{4}$$

where

$$[\sigma_+, \sigma_-] = \sigma_3 \qquad [\sigma_\pm, \sigma_3] = \mp 2\sigma_3 \tag{5}$$

and (σ_+, σ_-) represent the polarization. The major difference between the expressions of Hamiltonians (3) and (4) is that the two-level model has three independent variables (q-number) governed by the angular momentum commutation relations given by equations (5), which determine the nonlinearity of the corresponding dynamic equation [2]. However, the boson model has only two variables obeying the boson commutation relation, and the nonlinear term $\lambda b^+ b$ is responsible for the nonlinearity in the coupled photon–boson equation of motion. These lead to a difference between the two-level model and boson model, as far as dynamical behaviours of optical bistability and lasing action are concerned. The aim of this paper is to illustrate the lasing action arising from the nonlinear coupling between photons and bosons in contrast with the two-level model.

3. Coherent and incoherent injections

The system of photons and bosons mentioned above forms the central part of the laser (or bistable) operating device and hence cannot be isolated from its surroundings. First of all, the device should be able to supply coherent light power as an output, which is a kind of dissipation of the photon mode. In addition to this, there exists other photon mode dissipation (e.g. scattering, diffraction, non-intrinsic absorption etc). Moreover, other different dissipation mechanisms in the boson mode also exist. Obviously, in order to sustain the operation of the device some energy must be injected. There are two kinds of energy injection: coherent and incoherent. Coherent injection refers to the incident light, which is indispensable for passive bistable devices, whereas incoherent injection or the pump forms an essential part of the laser. Within the semiclassical approximation, after adding the phenomenological terms of dissipation, coherent injections, and incoherent injections mentioned above, the photon–boson coupling equations are derived from the Hamiltonian (equation (1)) as follows [8]:

$$\dot{\epsilon} = -(\gamma_\epsilon + \mathrm{i}(\omega_\epsilon - \omega))\epsilon - g(1 + \lambda|e|^2)e + \gamma_\epsilon \epsilon_i \tag{6}$$

$$\dot{e} = (\Gamma - \gamma_e - \mathrm{i}(\omega_e - \omega))e + g((1 + 2\lambda|e|^2)\epsilon - \lambda e^2 \epsilon^*) \tag{7}$$

where ϵ and e (and their complex conjugate ϵ^* and e^*) stand for the complex amplitudes of expected values of the photon operator a and the boson operator b (with their hermitian conjugate a^+ and b^+), respectively. That is

$$\langle a \rangle = \epsilon(t)\exp(-\mathrm{i}\omega t) \qquad \langle a^+ \rangle = \langle a \rangle^* \tag{8}$$

$$\langle b \rangle = e(t)\exp(-\mathrm{i}\omega t) \qquad \langle b^+ \rangle = \langle b \rangle^* \tag{9}$$

where ω represents the frequency of incident field ϵ_i. In the semiclassical approximation in which the correlation between various operators is neglected, the mean values of product can be factorized, for example,

$$n_\epsilon = \langle a^+ a \rangle = \langle a^+ \rangle \langle a \rangle = \epsilon^* \epsilon \tag{10}$$

$$n_e = \langle b^+ b \rangle = \langle b^+ \rangle \langle b \rangle = e^* e \tag{11}$$

where n_ϵ represents the quantum number of photon mode (the photon number in cavity), while n_e is the quantum number of the excited state of boson mode (the number of material bosons which indicates excitation energy level of the matter). On the other hand, the expected values of polarization operators (π, π^+) are

$$\begin{aligned} \langle \pi \rangle &= \langle (1 + \lambda b^+ b) b \rangle = (1 + \lambda \langle b^+ \rangle \langle b \rangle) \langle b \rangle \\ &= (1 + \lambda|e|^2) e \exp(-\mathrm{i}\omega t) \qquad \langle \pi^+ \rangle = \langle \pi^* \rangle. \end{aligned} \tag{12}$$

The factor $(1 + \lambda|e|^2)e$ and its complex conjugate are the amplitude of the polarization (P, P^*):

$$P = (1 + \lambda|e|^2)e \qquad P^* = (1 + \lambda|e|^2)e^*. \tag{13}$$

In equation (6), $\gamma_\epsilon \epsilon_i$ is the coherent driving term of the incident field ϵ_i, and γ_ϵ is the decay (dissipation) rate of photon mode. Moreover, when the term $(1 + \lambda|e|^2)e$ is interpreted as a matter polarization, equation (6) shows the typical form of Maxwell equation with slowly varying envelope and space average (mean field) approximation. In other words, it is exactly the same as the field equation coupled with the two-level atom matter equation (Maxwell–Bloch equation). Equation (7) and its complex conjugation as matter (boson) equations correspond to (but are of a different form from) Bloch equations of the two-level model. γ_e in equation (7) is the decay rate of the boson mode, whereas Γ represents the pumping rate corresponding to the incoherent energy injection to the material boson mode. In order to illustrate the dissipation and injection of energy in detail, we transform equations (6) and (7) into rate equations of the real amplitude and phase. Letting $\phi_\epsilon(\phi_e)$ be the phase of the photon (boson) mode relative to the incident light phase, which may be read as zero, we have

$$\epsilon = |\epsilon| \exp(\mathrm{i}\phi_\epsilon) = \sqrt{n_\epsilon} \exp(\mathrm{i}\phi_\epsilon) \qquad e = |e| \exp(\mathrm{i}\phi_e) = \sqrt{n_e} \exp(\mathrm{i}\phi_e). \quad (14)$$

Assuming the injected field ϵ_i is real, equations (6) and (7) become

$$\mathrm{d}n_\epsilon/\mathrm{d}t = -2\gamma_\epsilon n_\epsilon - 2g(1+\lambda n_e)\sqrt{n_e n_\epsilon}\cos(\phi_e - \phi_\epsilon) + 2\gamma_\epsilon \epsilon_i \sqrt{n_\epsilon}\cos\phi_\epsilon \quad (15)$$

$$\mathrm{d}n_e/\mathrm{d}t = -2\gamma_e n_e + 2g(1+\lambda n_e)\sqrt{n_e n_\epsilon}\cos(\phi_e - \phi_\epsilon) + 2\Gamma n_e \quad (16)$$

$$\mathrm{d}\phi_\epsilon/\mathrm{d}t = -(\omega_\epsilon - \omega) + g(1+\lambda n_e)\sqrt{n_e/n_\epsilon}\sin(\phi_\epsilon - \phi_e) + \gamma_\epsilon \epsilon_i/\sqrt{n_\epsilon}\sin\phi_\epsilon \quad (17)$$

$$\mathrm{d}\phi_e/\mathrm{d}t = -(\omega_e - \omega) + g(1+3\lambda n_e)\sqrt{n_\epsilon/n_e}\sin(\phi_\epsilon - \phi_e). \quad (18)$$

The rate equations (15) and (16) of quantum numbers n_ϵ and n_e, respectively, as real amplitude equations demonstrate the energy exchanges. The middle term on the right-hand side of equation (15) and that of equation (16) are equal in absolute value, but their signs are opposite. This term obviously represents the energy interchange between bosons and photons which arises from the bosons absorbing or emitting photons, which is the energy exchange inside the intrinsic system. The term $2\gamma_\epsilon n_\epsilon$ ($2\gamma_e n_e$) with negative sign stands for the rate of dissipative energy of photons (bosons) to its reservoir. The last term (symbolized simply by P_{coh}) on the right-hand side of equation (15),

$$P_{coh} = 2\gamma_\epsilon \epsilon_i |\epsilon| \cos\phi_\epsilon \qquad \left(|\epsilon| = \sqrt{n_\epsilon}\right) \quad (19)$$

is the work (per unit time) done by the incident light ϵ_i on the intrinsic system. The P_{coh}, depending on a definite phase ϕ_ϵ, i.e. the phase of photon mode relative to incident light, obviously stands for a coherent injection of energy. However, the counterpart in equation (16),

$$P_{incoh} = 2\Gamma n_e = 2\Gamma e e^* \quad (20)$$

is the power transferred directly to bosons, which is independent of the boson mode phase. The term $2\Gamma n_e$, just like the term $2\gamma_e n_e$, is the incoherent energy exchange. The $2\Gamma n_e$ is an 'input', while $-2\gamma_e n_e$ is an 'output'. The term $2\gamma_e n_e$ includes the spontaneous emission and the radiationless transition to a lower level of the boson mode, while $2\Gamma n_e$ is produced by the pump system exciting the bosons to higher energy level. All of these are illustrated in figure 1. The quantity Γ as a pumping rate plays an important role of control parameter, which will be shown later.

4. Stationary solution of laser equation and linear stability analysis

It is easy to show that when $\Gamma > \gamma_e$, the incident ϵ_i will be amplified by the stimulated emission of radiation of the boson mode. However, we are particularly interested in the lasing action without injected signal (i.e. the laser operation as a self-sustained oscillator) and its threshold behaviour; therefore, we omit the coherent driving term in equations (15) and (17), i.e. assume $\epsilon_i = 0$. Then, the reference frequency ω in equations (17) and (18) should be replaced by ω_ϵ or ω_e, respectively. Furthermore, our attention is focused on the case of pure resonance, i.e. $\omega = \omega_\epsilon = \omega_e$.

Without loss of generality, the phase of steady state ($\dot{n}_\epsilon = \dot{n}_e = \dot{\phi}_\epsilon = \dot{\phi}_e = 0$) of equations (15)–(18) on condition $\epsilon_i = 0$, $\omega = \omega_\epsilon = \omega_e$, and $g > 0$ becomes,

$$\sin(\phi_\epsilon - \phi_e) = 0 \qquad \text{and} \qquad \sin(\phi_\epsilon) = 0. \quad (21)$$

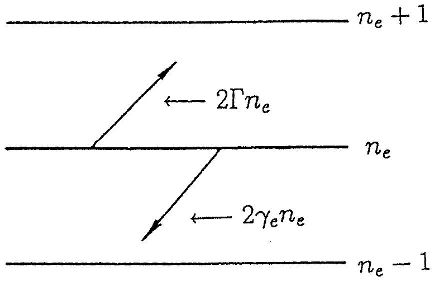

Figure 1. Illustration of the incoherent oscillator transition: pumping (transition to the higher level) and damping (transition to the lower level).

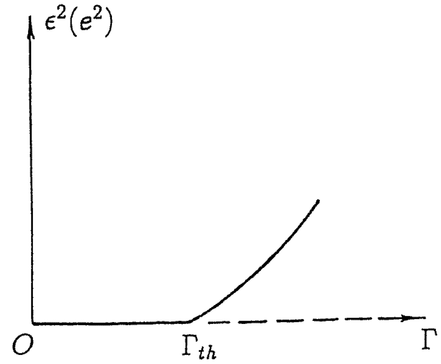

Figure 2. Light output (as a function of the matter excitation) $\epsilon^2(e^2)$ against pumping rate Γ curve: the full curve stands for the stable branch and the broken curve stands for the unstable branch.

The phase difference $\phi_\epsilon - \phi_e$ may be locked in this stationary value, so that ϵ and e become real numbers, i.e. $\epsilon = -|\epsilon| = -\sqrt{n_\epsilon}$, but $e = |e| = \sqrt{n_e}$, and then equations (6) and (7) reduces to the real equations

$$\dot{\epsilon} = -\gamma_\epsilon \epsilon - g(1 + \lambda e^2)e = I_\epsilon(\epsilon, e) \tag{22}$$

$$\dot{e} = (\Gamma - \gamma_e)e + g(1 + \lambda e^2)\epsilon = I_e(\epsilon, e) \tag{23}$$

which is a set laser equations for photon–boson coupling in the case of a single mode and for pure resonance. The steady-state equation may be expressed as

$$\epsilon_{st} = -g(1 + \lambda e_{st}^2)e_{st}/\gamma_\epsilon = -(\Gamma - \gamma_e)e_{st}/(g(1 + \lambda e_{st}^2)). \tag{24}$$

Obviously, there two different solutions of the steady states:

(i)

$$e_{st} = 0 \qquad \text{therefore } \epsilon_{st} = 0 \tag{25}$$

(ii)

$$e_{st}^2 = \frac{1}{\lambda}\left(\frac{\sqrt{(\Gamma - \gamma_e)\gamma_\epsilon}}{g} - 1\right) \qquad \epsilon_{st}^2 = \frac{\Gamma - \gamma_e}{\gamma_\epsilon \lambda}\left(\frac{\sqrt{(\Gamma - \gamma_e)\gamma_\epsilon}}{g} - 1\right). \tag{26}$$

Because e^2 $(= n_e)$ and ϵ^2 $(= n_\epsilon)$ must be taken as positive values, we have

$$\sqrt{(\Gamma - \gamma_e)\gamma_\epsilon}/g > 1 \qquad \text{or} \qquad \Gamma > g^2/\gamma_\epsilon + \gamma_e = \Gamma_{th}. \tag{27}$$

In other words, only when the pumping rate Γ is above a certain value Γ_{th}, laser action can occur. This is typical threshold behaviour of the laser.

However, we do not know whether the solution $\epsilon_{st} = 0$ ($e_{st} = 0$) exists when $\Gamma > \Gamma_{th}$. Therefore, in order to confirm the threshold behaviour mentioned above, we must proceed with a stability analysis of equations (22) and (23). Furthermore, through linear stability analysis the specific features of the boson model laser can be obtained. The results are summarized as follows

(i) The necessary stability conditions of the steady state $\epsilon = 0$ ($e = 0$) read

$$\text{(a)} \qquad \Gamma < \gamma_\epsilon + \gamma_e \tag{28}$$

$$\text{(b)} \qquad \Gamma < g^2/\gamma_\epsilon + \gamma_e = \Gamma_{th}. \tag{29}$$

(ii) The steady states $\epsilon^2 \neq 0$ ($e^2 \neq 0$) read

$$\text{(a)} \qquad \gamma_\epsilon + G - 2G_{th}\sqrt{G/G_{th}} > 0 \qquad (G = \Gamma - \gamma_e) \tag{30}$$

$$\text{(b)} \qquad \Gamma > g^2/\gamma_\epsilon + \gamma_e = \Gamma_{th}. \tag{31}$$

In equation (30) G is defined as a net gain. First of all, we would like to clarify the implication of condition (30). Assuming that Γ exceeds slightly the threshold, i.e.

$$\Gamma = \Gamma_{th} + \Delta \text{ or } G(= \Gamma_{th} + \gamma_e + \Delta) = G_{th} + \Delta \text{ but } \Delta \ll G_{th} \qquad (\Delta > 0). \tag{32}$$

We expand the square-root term of expression (30) as a series,

$$\sqrt{G/G_{th}} = \sqrt{1 + \Delta/G_{th}} = 1 + \frac{1}{2}(\Delta/G_{th}) - \frac{1}{2\times 4}(\Delta/G_{th})^2 + \cdots \tag{33}$$

so that equation (30) becomes

$$\gamma_\epsilon - G_{th} + \frac{1}{4}\Delta^2/G_{th} - \cdots > 0. \tag{34}$$

As Δ may be infinitesimal, this condition is equivalent to

$$\gamma_\epsilon > G_{th} = \Gamma_{th} - \gamma_e = g^2/\gamma_\epsilon \qquad \text{i.e. } \gamma_\epsilon > g \tag{35}$$

which shows that condition (28) is indeed superfluous.

The results of the steady-state solutions and the linear stability analysis are illustrated in figure 2. The $\epsilon^2(e^2)$ (laser light output) against Γ (pumping rate) curve is bifurcated at the threshold $\Gamma = \Gamma_{th}$: the full curve represents a stable branch, whereas the broken curve represents an unstable branch. Note that the threshold $\Gamma = \Gamma_{th}$ as a critical point loses the linear stability, since it cannot satisfy the linear stability condition (29) or (31) [14]. The present threshold as well as that of the two-level model laser indicates a continuous second-order phase transition, which is critically stable. This will be shown later.

Next, we discuss the threshold value Γ_{th} and the stability condition. From equations (27) and (35) we can see that the stability condition ($\gamma_\epsilon > g$) requires one to take a sufficiently high decay rate of the photon mode. Because $(\Gamma_{th} - \gamma_e)$ is inversely proportional to γ_ϵ, the high decay rate corresponds to the low pumping threshold Γ_{th}, which makes the laser start easily. On the other hand, the high photon mode decay rate means low reflectivity (high transitivity) at the output end of the cavity, so the external feedback by means of cavity mirrors is unnecessary for the laser and for the IAOB [4, 8] due to photon–boson coupling. This is an essential character which is different from the laser of the two-level model. The corresponding pump parameter of the two-level model laser is the unsaturated inversion of population D_0 and threshold $(D_0)_{th}$, which is expressed as [2]

$$(D_0)_{th} = \gamma_\epsilon \gamma_t / g'^2 \tag{36}$$

where g' is a coupling constant between photons and two-level atoms and γ_t is a transverse relaxation rate. Equation (36) shows that $(D_0)_{th}$ is directly proportional to the photon mode decay rate which is just opposite to the case of the boson model. Hence, the two-level model laser needs ‘good’ quality of cavity, but ‘bad’ quality of cavity is just suitable for the boson model laser.

5. Second-order phase transition

The system that is far from equilibrium, in which the Fokker–Planck equation satisfies the principle of detailed balance, possesses a so-called quasi-thermodynamic potential Ω involved in the stationary solution f in the following exponential form [13, 14]:

$$f \sim \exp(-2\Omega/Q) \tag{37}$$

where Q is a fluctuation constant. Generally, in the purely absorptive case and after a suitable adiabatic approximation the detailed balance will be satisfied by the nonlinear optical dissipative systems. We assume that with the condition of adiabatic approximation being $\gamma_\epsilon \gg \gamma_e$, which accords with the demands of the low pump threshold and the linear stability condition $\gamma_\epsilon > g$ (cf equation (35)), the photon mode ϵ may be treated as a fast variable slaved by the boson variable e. This is equivalent to setting (see equations (22) and (23))

$$\dot{\epsilon}(t \gg 1/\gamma_\epsilon) = -\gamma_\epsilon \epsilon - g(1 + \lambda e^2)e = 0 \tag{38}$$

so ϵ is eliminated from equation (23) and one obtains

$$\dot{e} = (\Gamma - \gamma_e)e - g^2(1 + \lambda e^2)^2 e/\gamma_\epsilon = I_e(\epsilon(e), e) \tag{39}$$

which is an equation of self-sustained oscillation. Here the factor $g^2(1 + \lambda e^2)^2 e/\gamma_\epsilon$ in the second term of the right-hand side should be interpreted as a rate of coherent radiation (stimulated emission) damping (the damping due to spontaneous emission is included in the decay rate γ_e). This coherent radiation damping reflects the energy transferring from boson mode to coherent photon mode. (cf also equations (15) and (16).)

It can be proved that the quasi-thermodynamic potential Ω involved in equation (37) is an integration of equation (39) as follows:

$$\Omega = -\int I_e(\epsilon(e), e)\,\mathrm{d}e = -\int ((\Gamma - \gamma_e)e - g^2(1 + \lambda e^2)^2 e/\gamma_\epsilon)\,\mathrm{d}e. \tag{40}$$

Neglecting the terms of more than sixth power in e, we can obtain the potential Ω near the threshold:

$$\Omega = \frac{1}{2}(G_{th} - G)e^2 + \frac{b}{4}e^4 \tag{41}$$

where

$$G_{th} = g^2/\gamma_\epsilon \qquad b = 2\lambda g^2/\gamma_\epsilon = 2g_{NL}g/\gamma_\epsilon. \tag{42}$$

The potential Ω gives a kind of stability criteria. If the steady state (either $e = 0$ or $e^2 \neq 0$) is stable, there must be

$$\partial^2\Omega/\partial e^2 = G_{th} - G + 3be^2 > 0 \qquad (b = 2g_{NL}g/\gamma_\epsilon > 0) \tag{43}$$

which is consistent with the results (29) and (31) of the linear stability analysis of equations (22) and (23) before adiabatic approximation. (However, after adiabatic approximation, $\gamma_\epsilon > g$, one of the linear stability conditions, is absent.) With the help of potential Ω we can also examine the stability of the critical point which loses the linear stability, i.e. at which

$$\partial^2\Omega/\partial e^2 = 0. \tag{44}$$

The stability of the critical point is determined by the third and fourth derivatives of Ω as follows [13]:

$$\partial^3\Omega/\partial e^3 = 0 \qquad \text{and} \qquad \partial^4\Omega/\partial e^4 (= 3!b) > 0. \tag{45}$$

Obviously, the point of the threshold ($G = G_{th}, e_{st} = 0$) is a stable critical point. The expression (40) shows the typical formulation of Landau mean field theory of second-order phase transition. Previously, on the basis of the two-level model Graham and Haken [11], Degiorgio and Scully [12] obtained separately the results similar to the formulation (41). To eliminate adiabatically the atom (matter) variables (polarization and population inversion) in the case of the two-level laser model, the matter (oscillator) variable e as an order parameter in the Ω formulation (41) should be replaced by the photon variable ϵ, so Ω is transformed as [1, 2]

$$\Omega = \frac{1}{2}(G_{th} - G)\epsilon^2 + \frac{b}{4}\epsilon^4 \tag{46}$$

where

$$G_{th} = g^2(D_0)_{th}/\gamma_t \qquad (G = g^2 D_0/\gamma_t) \qquad b = 4g^2 G_{th}/\gamma_t\gamma_l. \tag{47}$$

In spite of the great difference between two kinds of laser model in respect of the dynamics reflected by each Hamiltonian of the intrinsic system (cf equations (1) and (4)), the threshold behaviour of both lasers, shown separately in expressions (41) and (46), is basically the same. It demonstrates further that the critical phenomena and the corresponding phase transitions arising from different interactions have their significant generality. As a rule there is also no exception for the lasers in the systems that are far from equilibrium.

6. Discussion on nonlinear coupling

Nonlinear coupling has a significant effect on the lasing action due to the intrinsic interaction of light with material bosons as well as on IAOB. The intensity of this coupling is embodied in the parameter $g_{NL}(= \lambda g)$. Let equation (39) be rewritten as a rate equation of excitation energy level e^2 of matter (cf also (30) and (31)),

$$\frac{\mathrm{d}e^2}{\mathrm{d}t} = 2(G - G_{th})e^2 - 2g_{NL}e^4(2g + g_{NL}e^2)/\gamma_\epsilon \tag{48}$$

which is still the equation of self-sustained oscillation. The nonlinear coupling parameter as a coefficient of negative feedback, i.e. the term '$2g_{NL}e^4(2g + g_{NL}e^2)/\gamma_\epsilon$' with negative sign, ensures the stability of the operation. Certainly, it is easy to show that the variable of equation (48) can also be expressed in terms of the deviation δ of the e^2 from its stationary value $e_{st}^2(> 0)$, and the solution of δ with linear approximation under conditions $(G - G_{th}) \ll G_{th}(G > G_{th})$ and $|\delta| \ll e_{st}^2$ reads as

$$\delta(t) = \delta(0)\exp(-\Phi t) \qquad \delta(t) \equiv e^2(t) - e_{st}^2 \tag{49}$$

where the decay rate Φ of the deviation δ is just the potential stability criterion (43)

$$\Phi = \left(\frac{\partial^2\Omega}{\partial e^2}\right)_{e^2 = e_{st}^2 > 0} = -(G - G_{th}) + 6g_{NL}g e_{st}^2/\gamma_\epsilon \tag{50}$$

and

$$G - G_{th} \approx 2\sqrt{G_{th}}\left(\sqrt{G} - \sqrt{G_{th}}\right). \tag{51}$$

Taking the solution of $e_{st}^2 > 0$ (i.e. equation (26) where $\Gamma - \gamma_e = G$, $g^2/\gamma_\epsilon = G_{th}$ and $\lambda = g_{NL}/g$) into account, one obtains

$$\Phi = 4g_{NL}(g e_{st}^2/\gamma_\epsilon). \tag{52}$$

Obviously, the necessary stability condition of the steady $e_{st}^2 > 0$ is $\Phi > 0$ which means $g_{NL} > 0$, because we always have $g e_{st}^2/\gamma_\epsilon > 0$ in equation (52). Therefore, for the laser action of our model, the nonlinear coupling parameter g_{NL} should be positive. Furthermore, because $\gamma_\epsilon > g$ or $g/\gamma_\epsilon < 1$ (which is the stability condition (35)), the stability is ensured mainly by the parameter g_{NL}. Then the sufficiently large g_{NL} is also required in order to have corresponding ability against different kinds of fluctuation of the system. (The quantitative numbers of the g_{NL} would be decided by exact systems.) In fact, the condition $g_{NL} > 0$ is not only the stability condition of $e_{st}^2 > 0$ and $\epsilon_{st}^2 > 0$, but also the precondition of existing $e_{st}^2 > 0 (\epsilon_{st}^2 > 0)$, since both can be concluded as one condition:

$$G - G_{th} = 2g_{NL}(ge_{st}^2/\gamma_\epsilon) > 0. \tag{53}$$

The criteria (45), in which $3!b > 0$ means $g_{NL} > 0$, demonstrates that the nonlinear coupling constant g_{NL} is also related to the stability of the threshold (critical point $G = G_{th}$, $\epsilon_{st}(e_{st} = 0) = 0$).

7. Conclusion

(i) The nonlinear interaction of photons with the boson elementary excitation in the medium cannot only describe the mechanism for IAOB [8], but also provide an interesting laser model.
(ii) According to the demands of linear stability and the low threshold of pumping, the bosom model laser should have a considerable high decay rate of the photon mode. Therefore, external feedback by means of cavity mirrors is unnecessary for this laser model as well as for the IAOB which may be called boson model optical bistability [8, 9].
(iii) Though the dynamics of the boson model laser is different from the two-level model laser, in the vicinity of the critical point their generality becomes significant, since the threshold behaviour of the present laser model can also be cast into the framework of the Landau theory of second-order phase transition.

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